Problem : Crazy Cricket - Solution

Note that any n would be possible if Chris does not need to return to his own house, by following the path

$$1 \to n \to 2 \to n-1 \to 3 \to n-2 \to \dots.$$

The idea will be to modify this path slightly to fit Chris' needs. Unfortunately, this is not always possible, as some n do not admit a valid path. By trying some small n, it seems like $n \equiv 2, 3 \mod 4$ never work, whereas $n \equiv 0, 1 \mod 4$ always have a solution.

The key is to note that Chris can only travel back to his own house if he travels the same distance in both directions. In particular, the total distance traversed must be even in any solution. We know that Chris travels each distance in $\{1, 2, \dots, n-1\}$ exactly once, and hence the total distance must be

$$\sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1).$$

This is even if and only if $n \equiv 0, 1 \mod 4$, so that $n \equiv 2, 3 \mod 4$ indeed never works. If this is the case, we can thus output `impossible`. On the other hand, if $n \equiv 0, 1 \mod 4$, we do not know if it is possible until we find a construction that works.

We now modify the path given above to return to 1. Note that the path above takes steps of size $n-1, n-2, \dots$ in alternating directions, ending up somewhere in the middle of the street. The idea is to skip one of the step sizes (namely one close to $\frac{1}{2}n$) to use at the end to return from the middle of the street to 1. Skipping a single step size while alternating directions will only leave a single house unvisited, and as such cannot visit a single house multiple times (as we use $n-2$ steps, with which we visit $n-1$ unique houses including the start at 1).

The step size we skip is $\lfloor \frac{n}{2} \rfloor$. By distinguishing cases $n \equiv 0 \mod 4$ and $n \equiv 1 \mod 4$, it can be shown that in both cases, after alternating to step size 1, Chris ends up at house $\lfloor \frac{n}{2} \rfloor + 1$ and thus can jump back to 1 with the remaining step of size $\lfloor \frac{n}{2} \rfloor$. As an example, for $n = 4$ we get $1 \to 4 \to 3 \to 1$ and for $n = 5$ we get $1 \to 5 \to 2 \to 3 \to 1$. Hence, if $n \equiv 0, 1 \mod 4$ we simply alternate directions with decreasing step sizes, skipping $\lfloor \frac{n}{2} \rfloor$, and jumping back to 1 with the remaining jump.